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A nonlinear Landau-Zener model was proposed recently to describe, among a number of applications, the nonadiabatic transition of a Bose-Einstein condensate between Bloch bands. Numerical analysis revealed a striking phenomenon that tunneling occurs even in the adiabatic limit as the nonlinear parameter C is above a critical value equal to the gap V of avoided crossing of the two levels. In this paper, we present analytical results that give quantitative account of the breakdown of adiabaticity by mapping this quantum nonlinear model into a classical Josephson Hamiltonian. In the critical region, we find a power-law scaling of the nonadiabatic transition probability as a function of $C/V - 1$ and α , the crossing rate of the energy levels. In the subcritical regime, the transition probability still follows an exponential law but with the exponent changed by the nonlinear effect. For $C/V \gg 1$, we find a near unit probability for the transition between the adiabatic levels for all values of the crossing rate.

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I. INTRODUCTION

It is a common practice in the study of quantum systems to consider only a finite number of energy levels which are strongly coupled. The special case of truncation to two relevant levels is of enormous practical interest, and a vast amount of literature has been devoted to the dynamical properties of the two-level systems [1]. The Landau-Zener tunneling between energy levels is a basic physical process [2], and has wide applications in various systems, such as current driven Josephson junctions [3], atoms in accelerating optical lattices [4], and field-driven superlattices [5].

A nonlinear two-level system may arise in a mean field treatment of a many-body system where the particles predominantly occupy two energy levels, where the level energies depend on the occupation of the levels due to the interactions between the particles. Such a model arises in the study of the motion of a small polaron [6], for a Bose-Einstein condensate in a double-well potential [7–9] or in an optical lattice [10–12], or for a small capacitance Joseph-junction where the charging energy may be important. In contrast to the linear case, the dynamical property of a nonlinear two-level model is far from fully understood, and many novel features are revealed recently [13,14], including the discovery of a

nonzero Landau-Zener tunneling probability even in the adiabatic limit when the nonlinear parameter exceeds a critical value.

In this paper, we present a comprehensive theoretical analysis of the nonlinear Landau-Zener tunneling, obtaining analytical results for the tunneling behavior in various regimes. This is made possible by the observation that the population difference and the relative phase of the two levels form a pair of canonically conjugate variables of a classical Hamiltonian, the Josephson Hamiltonian [8,14]. The fixed points of this classical system correspond to the eigenstates of the nonlinear two-level Hamiltonian. Adiabatic evolution of the fixed points as a function of level bias correspond to adiabatic evolution of the two eigenstates. The breakdown of adiabaticity occurs when a fixed point turns into a homoclinic orbit which is possible if and only if the nonlinear parameter exceeds a critical value. Then the transition probability between the two energy levels can be nonzero even in the adiabatic limit, and is found to rise as a power law of the nonlinear parameter minus its critical value. Right at the critical value of the nonlinear parameter, the transition probability goes to zero as a power law of the crossing rate of the two levels. Below the critical point, the transition probability follows an exponential law as in the linear case but with the exponent changed due to the nonlinearity. Far above the critical point, we find a near unit probability of transition between the adiabatic levels for all values of the crossing rates.

Our paper is organized as follows. In Sec. II we introduce the nonlinear two-level model and discuss the behavior of its eigenstates and eigenenergies. In Sec. III, we cast the nonlinear two-level system into the Josephson Hamiltonian and study the evolution of the fixed points as functions of the system parameters. In Sec. IV we derive a formula for the tunneling probability in the adiabatic limit and analyze its power law behavior above and near the critical value of the nonlinear parameter. In Sec. V, we calculate the nonadiabatic tunneling probability in the critical and subcritical regimes. In Sec. VI, we derive the nonadiabatic tunneling probability for the ultrastrong nonlinear coupling using the stationary phase approximation. In sec. VII, we draw our conclusions and discuss how our findings may be observed experimentally.

II. THE NONLINEAR TWO-LEVEL MODEL

Our model system consists of two levels as in the standard Zener model but with an additional energy difference depending on the population in the levels. It is described by the following Hamiltonian matrix

$$H(\gamma) = \begin{pmatrix} \frac{\gamma}{2} + \frac{C}{2}(|b|^2 - |a|^2) & \frac{V}{2} \\ \frac{V}{2} & -\frac{\gamma}{2} - \frac{C}{2}(|b|^2 - |a|^2) \end{pmatrix}, \quad (1)$$

where a and b are the probability amplitudes. The Hamiltonian is characterized by three constants: the coupling between the two levels V , the level bias γ as in the linear Zener model, and the nonlinear parameter C describing the level energy dependence on the populations. The amplitudes satisfy the Schrödinger equation

$$i\frac{\partial}{\partial t} \begin{pmatrix} a \\ b \end{pmatrix} = H(\gamma) \begin{pmatrix} a \\ b \end{pmatrix}, \quad (2)$$

according to which, the total probability $|a|^2 + |b|^2$ is conserved and is set to be 1.

We wish to study how the system evolves when the level bias γ changes slowly from $-\infty$ to $+\infty$. It will be useful to find the adiabatic levels by diagonalizing the Hamiltonian (1), i.e., solving the following eigen-equations with an eigenenergy ϵ ,

$$\left(\frac{\gamma}{2} + \frac{C}{2}(|b|^2 - |a|^2)\right)a + \frac{V}{2}b = \epsilon a, \quad (3)$$

$$\frac{V}{2}a - \left(\frac{\gamma}{2} + \frac{C}{2}(|b|^2 - |a|^2)\right)b = \epsilon b, \quad (4)$$

For a solution with nonzero amplitudes, we impose the determinantal condition

$$\det \begin{pmatrix} \frac{\gamma}{2} + \frac{C}{2}(|b|^2 - |a|^2) - \epsilon & \frac{V}{2} \\ \frac{V}{2} & -\frac{\gamma}{2} - \frac{C}{2}(|b|^2 - |a|^2) - \epsilon \end{pmatrix} = 0, \quad (5)$$

which leads to a quartic algebraic equation for the eigenenergy,

$$\epsilon^4 + C\epsilon^3 + \left(\frac{C^2}{4} - \frac{V^2}{4} - \frac{\gamma^2}{4}\right)\epsilon^2 - \frac{V^2}{4}C\epsilon - \frac{V^2C^2}{16} = 0. \quad (6)$$

The eigenenergies ϵ correspond to the real roots of the above quartic equation. It is readily found that this quartic equation has two real roots when $C < V$, while there are four real roots when $C > V$. Two typical parameters are chosen to demonstrate the two cases in Fig.1. At $C/V = 0.5$ (Fig.1a), there are two energy levels corresponding to the two real roots; At $C/V = 2$ (Fig.1b), a loop appears at the tip of the lower level in the regime

$-\gamma_c \leq \gamma \leq \gamma_c$, where Eq. (6) has four real roots (this can be easily verified from the above equation at $\gamma = 0$). The relation between the singular point γ_c and the parameters will be given by Eq. (19) in the following section.

The eigenstates can be determined at the same time from the above equations,

$$a = \sqrt{\frac{1}{2} + \frac{\gamma}{2C + 4\epsilon}}, \quad (7)$$

$$b = \frac{V\sqrt{2C + 4\epsilon}}{4\epsilon\sqrt{C + 2\epsilon + \gamma}}. \quad (8)$$

These eigenstates are not orthogonal to each other for finite γ , but become so in the limits of $\gamma \rightarrow \pm\infty$, where $\epsilon \rightarrow \pm|\gamma|/2$. For instance, for the lower level, we have $(a, b) \rightarrow (1, 0)$ at $\gamma \rightarrow -\infty$ and $(a, b) \rightarrow (0, 1)$ at $\gamma \rightarrow +\infty$. Due to the nonlinearity of the model, there are two more eigenstates for $C > V$ and in the region $-\gamma_c < \gamma < \gamma_c$. These correspond to the loop structure in the adiabatic energy levels shown in Fig.1b.

III. THE CLASSICAL ANALOG

In this section, we cast the nonlinear two-level system into an effective classical system, i.e. the Josephson Hamiltonian. The dynamical mechanism for the breakdown of the adiabatic evolution will be revealed by investigating this classical Hamiltonian system. The Schrödinger equation (2) has the following explicit form

$$i\frac{\partial}{\partial t}a = \left(\frac{\gamma}{2} + \frac{C}{2}(|b|^2 - |a|^2)\right)a + \frac{V}{2}b, \quad (9)$$

$$i\frac{\partial}{\partial t}b = -\left(\frac{\gamma}{2} + \frac{C}{2}(|b|^2 - |a|^2)\right)b + \frac{V}{2}a, \quad (10)$$

which describes the evolution of the amplitudes for general occupations and phase of the wavefunctions. The physically interesting information of the state is completely represented by the population difference

$$s(t) = |b|^2 - |a|^2, \quad (11)$$

and the relative phase

$$\theta(t) = \theta_b(t) - \theta_a(t). \quad (12)$$

From Eqs. (9) and (10), we can obtain the following equations of motion

$$\frac{d}{dt}s(t) = -V\sqrt{1 - s^2(t)}\sin\theta(t), \quad (13)$$

$$\frac{d}{dt}\theta(t) = \gamma + Cs(t) + \frac{Vs(t)}{\sqrt{1 - s^2(t)}}\cos\theta(t). \quad (14)$$

These equations can be rewritten in the canonical form $\dot{s} = -\frac{\partial H_e}{\partial \theta}$, $\dot{\theta} = \frac{\partial H_e}{\partial s}$, with an effective classical Hamiltonian given by

$$H_e(s, \theta, \gamma) = \frac{C}{2}s^2(t) + \gamma s(t) - V\sqrt{1-s^2(t)}\cos\theta(t), \quad (15)$$

which has the form of a Josephson Hamiltonian.

Since we are interested in the situation of a quasistatic level bias γ , it will be useful to find the fixed points of the classical dynamics when γ is fixed in time. The fixed points correspond to the eigenstates of the nonlinear two-level system, and are obtained by equating the right hand sides of eqs.(13) and (14) to zero, yielding

$$\theta^* = 0, \pi, \quad (16)$$

$$\gamma + Cs^* + \frac{Vs^*}{\sqrt{1-s^{*2}}}\cos\theta^* = 0, \quad (17)$$

These can be consolidated into a single quartic equation

$$s^{*4} + 2\frac{\gamma}{C}s^{*3} - (1 - \frac{\gamma^2}{C^2} - \frac{V^2}{C^2})s^{*2} - 2\frac{\gamma}{C}s^* - \frac{\gamma^2}{C^2} = 0. \quad (18)$$

The number of the fixed points depends on the nonlinear parameters C . For weak nonlinearity, $C/V < 1$, only two fixed points exist, denoted as P_1 and P_2 in the panels of Fig.2. These are elliptic fixed points corresponding to the maximum and minimum of the classical Hamiltonian, and each are surrounded by closed (elliptic) orbits. The fixed points are located on the lines of $\theta^* = \pi$ and 0, meaning that the two corresponding eigenstates of the two level system have π and 0 relative phases. As the level bias changes from $\gamma = -\infty$ to $+\infty$, P_1 moves smoothly along the line $\theta^* = \pi$ from the bottom ($s = -1$) to the top ($s = +1$), corresponding to the lower branch of the energy levels in Fig.1(a). On the other hand, P_2 moves from the top to the bottom corresponding to the upper branch.

For stronger nonlinearity, $C/V > 1$, two more fixed points can appear when the level bias lies in a window $-\gamma_c < \gamma < \gamma_c$. The boundary of the window can be obtained from the condition of real roots for quartic equations, yielding

$$\gamma_c = (C^{2/3} - V^{2/3})^{3/2}. \quad (19)$$

As shown in Fig.3(c-e), both of the new fixed points lie on the line $\theta^* = \pi$, one being elliptic (P_4) as the original two and one being hyperbolic (P_3) corresponding to a saddle point of the classical Hamiltonian. One of the original fixed point, P_2 , still moves smoothly with γ , corresponding to the upper adiabatic level in Fig.1(b). The other, P_1 , moves smoothly up to $\gamma = \gamma_c$, where it collides

with P_3 , corresponding to the branch OXT of the lower level in Fig.1(b). The new elliptic point P_4 , created at $\gamma = -\gamma_c$ together with P_3 , moves up to the top, corresponding to the branch WXM of the lower level. The hyperbolic point P_3 , moves down away from its partner after creation and is annihilated again with P_1 at $\gamma = \gamma_c$, corresponding to the top branch WT of the lower level. The eventual fate of the fixed point P_1 will determine the adiabatic tunneling probability as shown below.

IV. ADIABATIC TUNNELING DUE TO NONLINEARITY

For quasistatic change of the level bias γ , a closed orbit in the classical dynamics remains closed such that the action

$$I = \frac{1}{2\pi} \oint sd\theta, \quad (20)$$

is invariant in time according to the classical adiabatic theorem [15], which is valid as long as the relative change of the system parameter in a period of the orbit is small, i.e.

$$Td\gamma/dt \ll \gamma. \quad (21)$$

The action equals the phase space area enclosed by the closed orbit, and is zero for a fixed point which has no area. Since the closed orbits surrounding an elliptic fixed point all have finite periods T , they should evolve adiabatically with the area of each fixed in time. We thus expect an elliptic fixed point to remain as a fixed point during the quasistatic change of the system parameter. For the case of $C/V < 1$, the two fixed points (both elliptic) evolve adiabatically throughout the entire course of change in the level bias, implying the absence of transition between the eigenstates in the adiabatic limit. This is still true for the fixed point P_2 in the case of $C/V > 1$, meaning a state starting from the upper level will remain in the upper level.

The adiabatic condition is broken, however, when P_1 collides with the hyperbolic fixed point P_3 to form a homoclinic orbit where the period T diverges. Nevertheless, the classical 'particle' will remain on this orbit, because the orbit is surrounded from both outside and inside by closed orbits of finite periods, which should form adiabatic barriers to prevent the particle from escaping. After this crisis, the homoclinic orbit turns into an ordinary closed orbit of finite period, and will evolve adiabatically for $\gamma > \gamma_c$ according to the rule of constant action which is now nonzero. In this region of the level bias, the population difference oscillates along the orbit, which is just the feature described in Ref. [13] that "there is a violent shaking above and about the lower branch of the adiabatic level after the terminal point".

The adiabatic tunneling probability can then be calculated from the action I_c of the homoclinic orbit. This orbit eventually evolves into a straight line at $s = s_f$ (see Fig.4), where the condition of constant action demands that

$$s_f = 1 - I(s_c). \quad (22)$$

On the other hand, from the definition of population difference, we find the final probabilities to be given by

$$\left(\frac{|a_f|^2}{|b_f|^2} \right) = \left(\frac{1-s_f}{1+s_f} \right). \quad (23)$$

The adiabatic tunneling probability is then expressed as

$$\Gamma_{ad} = |a_f|^2 = I_c/2. \quad (24)$$

The task we face then is to calculate the action of the homoclinic orbit. Let s_c be the population difference at the degenerate point P_c where P_1 collides with P_3 when $\gamma = \gamma_c$. It can be obtained by substituting the expression (19) into Eq. (18) as

$$s_c = -\frac{\sqrt{1-(V/C)^{2/3}}}{2} \left(1 + (V/C)^{2/3} \right) - \frac{\gamma_c}{2C}. \quad (25)$$

Considering the fact that the degenerate point P_c lies on $\theta = \pi$, we find the total energy (the value of the classical Hamiltonian) as

$$E_c = \frac{C}{2}(s_c)^2 + \gamma_c s_c + V\sqrt{1-(s_c)^2}. \quad (26)$$

The homoclinic orbit should have this energy, and its trajectory

$$s = s(\theta; E_c), \quad (27)$$

may be obtained by equating the classical Hamiltonian (15) to E_c . The corresponding action can then be evaluated to give the tunneling probability as

$$\Gamma_{ad} = \frac{1}{2}I(s_c) = \frac{1}{4\pi} \oint s(\theta; E_c) d\theta. \quad (28)$$

This has been evaluated numerically, and the results compare extremely well with those obtained by directly integrating the time-dependent nonlinear Schrödinger equation using the Runge-Kutta algorithm (See Fig.5).

The adiabatic tunneling probability can be evaluated analytically in the critical region of $\delta = C/V - 1 \rightarrow 0$. The singular point of the level bias is found to leading order as

$$\gamma_c \simeq V\left(\frac{2}{3}\delta\right)^{3/2}. \quad (29)$$

The homoclinic orbit is confined near the critical point, with its top at

$$s_t \simeq s_c + 3\sqrt{\frac{2}{3}}\delta. \quad (30)$$

We may expand the classical Hamiltonian to leading orders of $s - s_c$ and $\theta - \pi$ to find

$$\theta - \pi \simeq \sqrt{2\gamma_c(s - s_c)} + \frac{1}{2}\sqrt{2\gamma_c}(s - s_c)^{3/2}. \quad (31)$$

From the area of this orbit the adiabatic tunneling probability for this limiting case is found to be given by the power law

$$\Gamma_{ad} = \int_0^{s_t} (\theta - \pi) ds = \frac{4}{3\pi} \delta^{3/2}. \quad (32)$$

We may also obtain the adiabatic tunneling probability analytically in another limit, $C/V \rightarrow \infty$. To leading order in the small quantity $\sigma = V/C$, we find

$$\gamma_c \simeq C\left(1 - \frac{3}{2}\sigma^{2/3}\right), \quad (33)$$

$$s_c \simeq -1 + \frac{1}{2}\sigma^{2/3}. \quad (34)$$

In this case, the homoclinic orbit is confined near $s = -1$, and the deviation is the same order as $s_c - (-1)$. If we write $s - (-1) = \eta^2\sigma^{2/3}$, then

$$\frac{3}{4} - 3\eta^2 + \eta^4 - 2\sqrt{2}\eta\cos(\theta) \simeq 0. \quad (35)$$

By solving this equation and considering $s = \eta^2(\theta)\sigma^{2/3} - 1$, the adiabatic tunneling probability can be obtained

$$\Gamma_{ad} \simeq 1 - \frac{3}{2}\left(\frac{V}{C}\right)^{2/3}. \quad (36)$$

V. NONADIABATIC TUNNELING AT AND BELOW THE CRITICAL POINT

We have thus found that it is possible to tunnel between the adiabatic levels even in the adiabatic limit if the nonlinear parameter exceeds a critical value, i.e., $C/V > 1$. We now consider the case of nonzero sweeping rates, and study how the nonlinearity affects the nonadiabatic tunneling probability. In the linear case $C = 0$, the Landau-Zener formula prescribes an exponential tunneling probability between the adiabatic levels,

$$\Gamma \sim \exp\left(-\frac{\pi V^2}{2\alpha}\right). \quad (37)$$

For the nonadiabatic case, we will focus our attention in this section to the critical point and the subcritical region, and consider the near adiabatic case (i.e., $\alpha \neq 0$ and $\alpha \ll 1$).

For this purpose, we need to investigate the evolution of the fixed point P_1 as well as the nearby periodic orbits. Then, in addition to the action variable I mentioned in the above section, we should introduce its canonical conjugate – the angle variable ϕ . They satisfy the following differential equations [15],

$$\dot{I} = -\frac{\partial R}{\partial \phi} \dot{\gamma}, \quad (38)$$

$$\dot{\phi} = \omega(I, \gamma) + \frac{\partial R}{\partial I} \dot{\gamma}. \quad (39)$$

The function $R(I, \phi)$ is defined as $\frac{\partial}{\partial \gamma} \int s(\theta, E, \gamma) d\theta$, where E is the energy of the periodic orbit. Here, ω is the frequency of orbits in the immediate neighborhood of the fixed point P_1 . Its expression is particularly simple at $I = 0$, corresponding to the fixed points P_1 . It can be calculated by linearizing the equations of motion eqs.(13,14) near the fixed point as

$$\omega^* = V \sqrt{1/(1 - (s^*)^2) - \frac{C}{V} \sqrt{1 - (s^*)^2}}. \quad (40)$$

As in the adiabatic case considered in the last section, the transition probability is still given by the increment of the action, i.e.,

$$\Gamma = \frac{1}{2} \Delta I. \quad (41)$$

The reason is as follows. The initial state at $\gamma = -\infty$ is the fixed point P_1 with zero action, and the final state is an orbit of finite action which becomes a horizontal line $s = s_f$ at $\gamma = +\infty$. Then an integration of Eq.(38) yields

$$\Delta I = - \int_{-\infty}^{+\infty} \frac{\partial R}{\partial \phi} \frac{d\gamma}{dt} \frac{d\phi}{\dot{\phi}}. \quad (42)$$

To evaluate this integral, we need to express $\dot{\phi}$ as a function of ϕ itself. Near the adiabatic limit, we may omit the second term in Eq.(39) and set $\omega(I, \gamma) = \omega^*(\gamma)$ in the first term, yielding

$$\dot{\phi} = V \sqrt{1/(1 - (s^*)^2) - \frac{C}{V} \sqrt{1 - (s^*)^2}}. \quad (43)$$

On the other hand, by substituting $\theta^* = \pi$ into eq.(17) and differentiating with respect to time, one gets

$$\frac{dt}{ds^*} = \frac{V}{\alpha} \left(1/\sqrt{1 - (s^*)^2} + (s^*)^2/(1 - (s^*)^2)^{3/2} - C/V \right). \quad (44)$$

Combining these equations, one may relate s^* to ϕ and thus express $\dot{\phi}$ as a function of ϕ itself.

The principal contribution to the integral comes from the neighborhood of the singularities of the integrand,

which comes only from the zeros of the frequency $\dot{\phi} = \omega^*(\gamma)$. These zero points are easily found from Eq.(40) as

$$s_0^* = [1 - (V/C)^{2/3}]^{1/2}. \quad (45)$$

As will be shown below, the integral is exponentially small if there is no real singularities, and becomes a power law in the sweeping rate if there is a singularity on the real axis.

We first consider the case of critical nonlinearity, $C/V = 1$, for which the singular point occurs at $s^* = 0$. Near this point, we find from Eqs.(44) that $\omega^* \simeq \sqrt{\frac{3}{2}} V s^*$ and $\phi \simeq \frac{1}{4} (\frac{3}{2})^{3/2} \frac{V^2}{\alpha} (s^*)^4$. Then, we have an approximate relation $\omega^* \sim \alpha^{1/4} \phi^{1/4}$ near the singularity. Substituting these expressions back to the eq.(42), and utilizing the fact that $\frac{\partial R}{\partial \phi}$ is independent of α , we find a power-law behavior for the tunneling probability

$$\Gamma \sim \alpha^{3/4}. \quad (46)$$

This scaling law has been verified by our numerical calculations (Fig.6).

Now we extend our discussion to nonadiabatic tunneling for subcritical nonlinearity, where the zeros of the frequency ω^* are complex. Because R is a periodic function of the angle variable, it can be expanded as a Fourier series,

$$\frac{\partial R}{\partial \phi} = \sum_{l=-\infty}^{+\infty} i l e^{i l \phi} R_l. \quad (47)$$

After substituting this series into eq.(42), we may deform the contour of integration into the complex plane. For those positive l terms in the series, we may raise the contour into the upper half plane until it is "caught up" by a singularity of the integrand, namely a zero of ω^* . Let ϕ_0 be the singularity nearest to the real axis, i.e. the one with the smallest positive imaginary part. The principal contribution to the integral comes from the neighborhood of this point, yielding [15],

$$\Delta I \sim \exp(-l_s \text{Im}(\phi_0)), \quad (48)$$

where 'Im(ϕ_0)' indicates the imaginary part of the complex number ϕ_0 and l_s is the lowest order of the Fourier series whose Fourier component does not vanish. The negative l terms give the same exponential and so affects only the prefactor which we do not consider here. In the neighborhood of the elliptic fixed point P_1 , the system can be approximated by a harmonic oscillation with the function $R \sim \sin(2\phi)$, so we have $l_s = 2$.

To find the complex zero ϕ_0 , we combine the equations of (44) to express it as an integral of s^*

$$\phi_0 = \frac{V^2}{\alpha} \int_0^{s_0^*} \sqrt{1/(1 - (s^*)^2) - \frac{C}{V} \sqrt{1 - (s^*)^2}}$$

$$\left(\frac{1}{\sqrt{1 - (s^*)^2}} + (s^*)^2 / (1 - (s^*)^2)^{3/2} - C/V \right) ds^*. \quad (49)$$

We take the upper bound to be the zero s_0^* of the frequency, while the lower bound only reflects a certain choice of the origin of ϕ which does not affect the imaginary part.

The tunneling probability is thus found to be proportional to the exponential

$$\Gamma \sim \exp(-q \frac{\pi V^2}{2\alpha}), \quad (50)$$

where the factor in the exponent is given by

$$q = \frac{4}{\pi} \int_0^{\sqrt{(V/C)^{2/3} - 1}} (1+x^2)^{1/4} (1/(1+x^2)^{3/2} - C/V)^{3/2} dx. \quad (51)$$

For the linear case $C = 0$, the factor q is exactly unit, which means that the expression of the tunneling probability is identical to the standard Landau-Zener formula. For the nonlinear case, $C/V > 0$, this factor becomes smaller than one, indicating the enhancement effect on the nonadiabatic tunneling. As C/V goes up to 1, the critical point, this factor vanishes, meaning the breakdown of the exponential law.

In Fig.7, we plot this factor and compare it with the results of numerical integration of the nonlinear Schrödinger equation. The agreement is very well for the nearly linear case, $C/V \ll 1$, and there is some deviation when C/V is close to unity. One reason for the latter is that the prefactor may be important in such a case. Also, numerical accuracy may be blamed. To obtain the correct slope in the plot of $\ln \Gamma$ vs. α , one should calculate the tunneling probability at infinitesimal α . However, in practical calculations, this parameter is chosen as a finite value instead of infinitesimal. In [13], the slope is read in the regime $\alpha \in [0.02, 0.01]$. In our calculations, this range is extended to $[0.005, 0.0025]$. As is well known, in the regime of small α , the numerical fluctuation is serious. This may result in a relative large slope numerically.

VI. TUNNELING AT STRONG NONLINEARITY

In an earlier section, we have found that for strong nonlinearity $C/V \gg 1$, there is a near unity tunneling probability to the upper adiabatic level even in the adiabatic limit. This probability can only gets larger when the sweeping rate is finite. We thus expect that the amplitude b in the original Schrödinger equation remains small all the times, and a perturbative treatment of the

problem becomes adequate. We begin with the variable transformation

$$a = a' \exp \left(-i \int_0^t \left(\frac{\gamma}{2} + \frac{C}{2} (|b|^2 - |a|^2) \right) dt \right), \quad (52)$$

$$b = b' \exp \left(i \int_0^t \left(\frac{\gamma}{2} + \frac{C}{2} (|b|^2 - |a|^2) \right) dt \right). \quad (53)$$

Then the diagonal terms in Eqs. (9,10) are transformed away, and our above discussions imply that we can take $a' \approx 1$, and

$$b' = \frac{V}{2i} \int_{-\infty}^t dt \exp \left(-i \int_0^t (\gamma + C(|b|^2 - |a|^2)) dt \right). \quad (54)$$

We need to evaluate the above integral self-consistently. Because of the large C , the nonlinear term in the exponent generally gives a rapid phase change which makes the integral small. The dominant contribution comes from the stationary point t_0 of the phase where $(-\gamma + C(1 - 2|b|^2))_{t_0} = 0$. Expanding about this point, we may write

$$-\gamma + C(1 - 2|b|^2) = -\bar{\alpha}(t - t_0), \quad (55)$$

with

$$\bar{\alpha} = \alpha + 2C \left[\frac{d}{dt} |b|^2 \right]_{t_0}. \quad (56)$$

We thus have

$$|b|^2 = \left(\frac{V}{2} \right)^2 \left| \int_{-\infty}^t dt \exp \left(-\frac{i}{2} \bar{\alpha} (t - t_0)^2 \right) \right|^2. \quad (57)$$

We can differentiate this expression and evaluate its result at time t_0 , obtaining a few standard Fresnel integrals with the result $[\frac{d}{dt} |b|^2]_{t_0} = \left(\frac{V}{2} \right)^2 \sqrt{\frac{\pi}{\bar{\alpha}}}$. Combining this with the relation (56), we come to a closed equation for $\bar{\alpha}$,

$$\bar{\alpha} = \alpha + 2C \left(\frac{V}{2} \right)^2 \sqrt{\frac{\pi}{\bar{\alpha}}}. \quad (58)$$

The nonadiabatic transition probability Γ is given by

$$\begin{aligned} \Gamma = 1 - |b|_{+\infty}^2 &= 1 - \left(\frac{V}{2} \right)^2 \left| \int_{-\infty}^{+\infty} dt \exp \left(-\frac{i}{2} \bar{\alpha} (t - t_0)^2 \right) \right|^2 \\ &= 1 - \frac{\pi V^2}{2\bar{\alpha}}. \end{aligned} \quad (59)$$

Then the above result yields a closed equation for the Γ ,

$$\frac{1}{1 - \Gamma} = \frac{1}{P} + \frac{\sqrt{2} C}{\pi V} \sqrt{1 - \Gamma}, \quad (60)$$

where $P = \frac{\pi V^2}{2\alpha}$. In the adiabatic limit, i.e. $\frac{1}{P} = 0$, we find that $\Gamma = 1 - 1.7 \left(\frac{V}{C} \right)^{\frac{2}{3}}$. This gives the same exponent as the exact asymptotic result (36) obtained in an earlier section, but the coefficient differs by about 10%. In the sudden limit, $\frac{1}{P} \rightarrow \infty$, we have $\Gamma = 1 - P$, which is exact. In figure 8, we compare the above analytical results with that from directly solving the Schrödinger equation and show a good agreement.

In this paper, we present a theoretical analysis on the Landau-Zener tunneling in a nonlinear two-level system. Our results can be summarized as: 1) The adiabatic tunneling probability depends only on the ratio between the nonlinear parameter C and the coupling constant V . With increasing the ratio C/V , a transition from zero adiabatic tunneling probability to nonzero adiabatic tunneling probability occurs at a critical point $C/V = 1$. The critical behavior near the transition point follows a power-law scaling. 2) In the subcritical regime, the nonadiabatic tunneling probability follows an exponential law, which modifies the Landau-Zener formula by adding a factor in the exponent. This exponential law breaks down at the critical point, instead, a power-law shows up. 3) Far above the critical point, we find a near unit probability of transition between the adiabatic levels for all values of the crossing rates. The analytical expression is obtained using the stationary phase approximation.

The experimental observation of these findings is a topic of great interest. It has been shown that the nonlinear two-level model can be used to understand the tunneling of a Bose-Einstein condensate between Bloch bands in an optical lattice [13]. Another physical model for direct application of our theory is a Bose-Einstein condensate in a double-well potential [7,8,16]. The amplitudes of general occupations $N_{1,2}(t)$ and phases $\theta_{1,2}$ obey the nonlinear two-mode Schrödinger equations similar to (2) (e.g. see [16]). After introducing the new variables $s(t) = (N_2(t) - N_1(t))/N_T$ and $\theta = \theta_2 - \theta_1$, one also obtain a Hamiltonian having the same form as Eq.(15) except for the parameters replaced by $V = 2K/\hbar, \gamma = -[(E_1^0 - E_2^0) - (U_1 - U_2)N_T/2]/\hbar, C = (U_1 + U_2)N_T/2\hbar$. Here the constants have following meanings: Total number of atoms $N_1 + N_2 = N_T$, is conserved, $E_{1,2}^0$ are zero-point energy in each well, $U_{1,2}N_{1,2}$ are proportional to the atomic self-interaction energy, and K describes the amplitude of the tunneling between the condensates.

In practice, such a double well may be achieved by a laser sheet dividing a trap, and the energy levels may be moved by shifting the laser sheet. Then, our theoretical results can be directly applied to this system without intrinsic difficulty. One can design an experiment for observing the transition to nonzero adiabatic tunneling and verify those scaling law found in our paper. We hope our discussions will stimulate the experimental works in the direction.

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Figure Captions

Fig.1 Adiabatic energy levels for two typical nonlinear cases (solid line). The dashed lines are the ones for linear case ($C = 0$). P_i ($i = 1, \dots, 4$) is the fixed points of H_e system, corresponding to the quantum eigenstates as shown in (b): $OXT \rightarrow P_1$, $MXW \rightarrow P_4$, $WT \rightarrow P_3$, only P_3 is an unstable saddle point, others are stable elliptic point.

Fig.2 The phase-space portrait of the Hamilton H_e system for various parameter γ at $C/V = 0.5$. The arrows refer to the direction of the shift of the fixed points P_i as γ increases. The closed curves are the periodic trajectories. In this case, no collision between fixed points occurs, which indicates a zero adiabatic tunneling probability.

Fig.3 The phase-space portrait of the Hamilton H_e system for various parameter γ at $C/V = 2$. The arrows refer to the direction of the shift of the fixed points as γ increases. In this case, a collision occurs at a singular point γ_c , where P_2 and P_3 collide and form a homoclinic orbit with nonzero action. This jump on the action leads to a nonzero adiabatic tunneling probability.

Fig.4 The homoclinic trajectories associated with the singular points P_c at critical point $\gamma = \gamma_c$ (left column), and its final shape at $\gamma = +\infty$ after adiabatic evolution (right column), respectively. For the action keeps as an invariant during the evolution process, the areas shadowed are equal for the same C/V . Here we show two typical situations corresponding to smaller and larger nonlinearity, respectively.

Fig. 5 The adiabatic tunneling probability vs. parameter C/V . Solid line is our theoretical results, whereas the symbols—dot, star and plus denote the results from solving nonlinear Schrödinger equation numerically. Inset is a magnification of the local plot.

Fig.6 The dependence of the tunneling probability on the parameter α for the critical nonlinearity.

Fig.7 The dependence of the factor q on the ratio C/V .

Fig.8 The comparison between our analytical theory and the numerical simulations in the regime of strong nonlinear coupling.

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- [1] For example see review, M. Grifoni and P. Hanggi, Phys. Rep. **304** (1998) 229-354, and references therein.
 - [2] L.D. Landau, Phys. Z. Sowjetunion **2**, 46 (1932); G. Zener, Proc. R. Soc. London, Ser. A **137**, 696 (1932).
 - [3] K. Mullen, E. Ben-Jacob, and Z. Schuss, Phys. Rev. Lett. **60**, 1097 (1988); K. Mullen et al., Physica B **153**, 172 (1988); U. Greigenmüller and G. Schön, ibid. **153**, 186 (1988).
 - [4] C.F. Bharucha, K.W. Madison, P.R. Morrow, S.R. Wilkinson, B. Sundaram, and M.G. Raizen, Phys. Rev. A **55**, R857 (1997); K.W. Madison, C.F. Bharucha, P.R.

- Morrow, S.R. Wilkinson, Q. Niu, B. Sundaram, and M.G. Raizen,, Appl. Phys. B; Lasers Opt. **65**, 693 (1997).
- [5] A. Sibille, J.F. Palmier, and F. Laruelle, Phys. Rev. Lett. **80**, 4506 (1998)
 - [6] J.C. Eilbeck, P.S. Lomdahl, and A.C. Scott, Physica D **16**, 318 (1985); V.M. Kenkre and D.K. Campbell, Phys. Rev. B **34**, 4959 (1986); P.K. Datta and K. Kundu, *ibid.* **53**, 14929 (1996)
 - [7] M.R. Andrews, C.G. Townsend, H.-J. Miesner, D.S. Durfee, D.M. Kurn, and W. Ketterle, Science **275**, 637 (1997)
 - [8] A. Smerzi, S. Fantoni, S. Giovannazzi, and S.R. Shenoy, Phys. Rev. Lett. **79**, 4950 (1997); G.J. Milburn et al. Phys. Rev. A **55**, 4318 (1997)
 - [9] M.O. Mewes, M.R. Andrews, D.M. Kurn, D.S. Durfee, C.G. Townsend, and W. Ketterle, Phys. Rev. Lett. **78**, 582 (1997)
 - [10] M.H. Anderson, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Science **269**, 198 (1995)
 - [11] Dae-Il Choi and Q. Niu, Phys. Rev. Lett. **82**, 2022 (1999)
 - [12] W. Zhang and D.F. Walls, Phys. Rev. A **49**, 3799 (1994); W. Zhang and D.F. Walls, *ibid.* **52**, 4696 (1995)
 - [13] B.Wu and Q.Niu, Phys. Rev. A **61** (2000) 023402
 - [14] O.Zobay and B.M.Garraway, Phys. Rev. A **61** (2000) 033603
 - [15] L. D. Landau and E. M. Lifshitz, *Mechanics*. (Rergamon, Oxford, 1977)
 - [16] S.Raghavan, A.Smerzi, S.Fantoni, and S.R.Shenoy, Phys. Rev. A **59**, 620 (1999)

















